The formulas for the empirical terms of other elements can be obtained analogously.

In conclusion we would like to express deep gratitude to Yu. V. Batrakov for valuable advice relating to the present work.

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Decay of Shock Waves in Stationary Flows

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THE asymptotic laws of extinction of nonstationary shock waves at great distances frôm the point of explosion were found by Landau,[1] Khristianovich,[2] Sedov,[3] and Whitham[4] on the assumption that the medium is homogeneous (uniform). A study of the propagation of shock waves in a nonuniform moving medium in the realm of geometrical acoustics was carried out in the work of Keller.[5] Articles by Gubkin,[6] Otterman,[7] Polianskii,[8] and the author[9] are devoted to a refinement of the acoustical theory in which the nonlinear nature of the gasdynamic equations is not taken into account.

The laws of extinction of weak shock waves in steady-state supersonic flows have been studied less. They were established by Landau[1] and Whitham[10] for plane-parallel and axially symmetric uniform flows. The behavior of plane shock waves was studied in the work of Friedrichs[11] with a higher degree of accuracy in second approximation.

The present work explains the main features of the development of shock waves of small amplitude in nonuniform steady-state supersonic flows; the width of the perturbed region of flow is assumed small as compared with the radius of curvature of the density jump and with the distance at which the parameters of the initial medium change essentially. On the basis of the investigation, it was proposed that each small element of the perturbed region in first approximation may be examined as a nonstationary Rieman wave carried laterally by the uniform flow. The laws of the variation of the parameters of the medium at the shock fronts may be then used with the method similar to that used by Landau.[1] It appears, however, that the results obtained in Ref. 9 for nonstationary shock waves moving in a nonuniform medium cannot be applied directly for a computation of steady-state supersonic flows. The cause for this is included in the different laws determining the variation of the width of the perturbed region in steady-state and nonstationary processes.

1. The initial system of gasdynamic equations may be written

$$v_j \frac{\partial v_i}{\partial x_j} + \frac{1}{\rho}\frac{\partial p}{\partial x_i} = g_i \qquad \frac{\partial \rho v_j}{\partial x_j} = 0$$
$$v_j(\partial s/\partial x_j) = 0 \qquad p = p(\rho, s) \tag{1.1}$$

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Here v_i, g_i, p, ρ, and s designate the velocity components of the flow and of the mass forces, pressure, density, and entropy at the point with the rectangular coordinates x_i, respectively. The usual tensor notation of the sums is used according to the repeating subscripts i, j, which take on the values 1, 2, 3.

Subsequently, we shall examine only supersonic flows. The required system of equations (1.1) will then be of the hyperbolic type. The equation which determines the C_+-characteristic surfaces $\varphi(x_i) = 0$ of this system, may therefore be written

$$v_j n_j + a = 0 \qquad a = \sqrt{(\partial p/\partial \rho)_s} \tag{1.2}$$

Here a is the speed of sound and n_i the components of the normal to these surfaces, for which the following formula holds:

$$n_i = \frac{\partial\varphi/\partial x_i}{\sqrt{(\partial\varphi/\partial x_j)^2}}$$

The system of gasdynamic equations (1.1) on the C_+-characteristics takes on the form

$$(v_i + an_i)\frac{\partial p}{\partial x_i} + a\rho(a\delta_{ij} + n_i v_j)\frac{\partial v_i}{\partial x_j} = \rho a n_i g_i$$
$$\delta_{ij} = 1 \text{ when } i = j \qquad \delta_{ij} = 0 \text{ when } i \neq j \tag{1.3}$$

Equation (1.3) contains the derivatives of the sought functions only along the C_+-characteristic surfaces.

We proceed to a study of the behavior of weak shock waves in a nonuniform flow. We consider that, in an unperturbed state, the pressure p_0, the density ρ_0, the speed of sound a_0, and the velocity components of the flow v_{0i} are given as functions of the coordinates x_i. Because of the smallness of the amplitude of the density jump, the relative variations of all the gas parameters at its front are small. Therefore, we assume

$$p = p_0 + p' \qquad \rho = \rho_0 + \rho'$$
$$a = a_0 + a' \qquad v_i = v_{0i} + v_i'$$

Here p', ρ', a', and v_i' are the excess pressure, density, speed of sound, and velocity components of the particles in the region of perturbed flow. The dimensionless values

p'/p_0, ρ'/ρ_0, a'/a_0, and v_i'/a_0 are small as compared with unity; therefore, their squares will be disregarded later on. We make two more assumptions with respect to the geometric properties of the investigated flows; namely: we consider that the region of perturbed motion of the gas is narrow, i.e., its width λ_* is much less than the characteristic radius of curvature of the shock front R and the distance H, at which the parameters of the medium in a state of equilibrium vary substantially. We shall also disregard the values λ_*/R and λ_*/H as compared with unity. We assume, furthermore, that in directions tangent to the front of the density jump, all the excess values vary slowly (i.e., the variations are assumed to be significant at distances of the order of R and H).

In acoustical approximation, we can identify a shock wave with the C_+-characteristic surface (1.2), whose equation assumes the form

$$v_{0j}n_j + a_0 = 0 \tag{1.4}$$

as a result of neglecting the small values.

The characteristic curves (bicharacteristics) of Eq. (1.4) are the trajectories of the elements of the surface of the shock front or the rays; they are determined by a system of ordinary differential equations

$$\frac{dx_i}{v_{0i} + a_0 n_i} = \frac{dn_i}{(n_i n_j - \delta_{ij})(\partial a_0/\partial x_j + n_k \partial v_{0k}/\partial x_j)} \tag{1.5}$$

In steady-state supersonic flow, each element of the shock wave surface in its movement along the ray passes through the trajectory of the previous element in a manner similar to any gas particle moving along the line of flow; it repeats the path of the previous particle. We note that in the case of steady-state flows the rays lie totally on the characteristic surfaces $\varphi(x_i) = 0$, whereas in nonstationary processes the rays in the space (x_i) intersect the wave fronts $\varphi(t, x_i) = 0$, nowhere touching these surfaces. This fact, as will be shown later on, leads to different laws of extinction of shock waves in steady-state and nonstationary processes.

The following relationships between the perturbed gas parameters hold in first approximation at shock fronts of small amplitude:

$$\rho' = \frac{1}{a_0^2}p' \qquad a' = \frac{m_0 - 1}{\rho_0 a_0}p' \qquad v' = \frac{1}{\rho_0 a_0}p'$$
$$m_0 = \frac{1}{2\rho_0^3 a_0^2}\left(\frac{\partial^2 p}{\partial V_0^2}\right)_3 \qquad V = 1/\rho' \qquad v' = \sqrt{(v_i')^2} \tag{1.6}$$

wherein $v_i' = v'n_i$. For an ideal gas the coefficient m_0 is equal to $(\kappa + 1)/2$ where κ is the Poisson adiabatic index. Since in the examined approximation the shock waves coincide with the characteristic surfaces, formulas (1.6) may be used to eliminate the functions ρ', a', and v_i' from Eq. (1.3) after its linearization. As a result, we have

$$\frac{1}{p'}\frac{dp'}{dt} - \frac{1}{2\rho_0 a_0}\frac{d\rho_0 a_0}{dt} + \frac{1}{2}\left(a_0\frac{\partial n_j}{\partial x_j} + k_0\frac{\partial v_{0j}}{\partial x_j} + n_i n_j\frac{\partial v_{0i}}{\partial x_j}\right) = 0 \tag{1.7}$$
$$k_0 = 2m_0 - 1$$

Here

$$\frac{d}{dt} = (v_{0j} + a_0 n_j)\frac{\partial}{\partial x_j}$$

denotes the derivative along the ray, determined by system (1.5). For an ideal gas, the coefficient k_0 is equal to the Poisson adiabatic index κ. Integrating Eq. (1.7), we obtain the Keller formula that gives the law of variation of the excess pressure at the front of the density jump[5]:

$$p' = \frac{p_0'}{L}\sqrt{\frac{\rho_0 a_0}{\rho_{00} a_{00}}}$$
$$L = \exp\left[\frac{1}{2}\int_{t_0}^{t}\left(a_0\frac{\partial n_j}{\partial x_j} + k_0\frac{\partial v_{0j}}{\partial x_j} + n_i n_j\frac{\partial v_{0i}}{\partial x_j}\right)dt\right] \tag{1.8}$$

Here p_0', ρ_{00}', and a_{00} denote the excess pressure at the shock front and the equilibrium density and speed of sound at the initial time t_0, taken at the initial point of the ray. Since the value of the ray velocity $q_0 = \sqrt{(v_{0j} + a_0 n_j)^2}$ will be $\sqrt{v_0^2 - a_0^2}$, the time t is associated with the length of the ray l by the relationship

$$dt = \frac{1}{\sqrt{v_0^2 - a_0^2}}dl \qquad v_0 = \sqrt{(v_{0i})^2} \tag{1.9}$$

Formula (1.8) is correct both for steady-state and for nonstationary processes of shock wave propagation.

According to hypothesis, the width of the perturbed region λ_* which analogously with the nonstationary flows will also be called the wavelength, is small as compared with the characteristic dimensions R and H, whereas the excess values p', ρ', a', and v_i' change essentially in directions tangent to the shock front only in distances of the same order as R and H. It was shown in Ref. 6 that, in such a case, formulas (1.6) and (1.8) are correct in the entire field of the perturbed flow, if we understand p_0' to mean the excess pressure, taken at the arbitrary point of the wave profile when $t = t_0$.

We denote the wavelength by λ in acoustical approximation. We derive the law of variation of λ when the wave element is moving along the ray. To do this, it is convenient to examine the structure of the perturbed region in steady-state motion of the gas from the point of view of the nonstationary flows.

We take an arbitrary point on the characteristic surface which has the value zero of the excess pressure and surround it with a small cylinder with an axis which coincides with the normal n_{0i} to this surface. As the bases of the cylinder, we take the characteristic surface, where $p' = 0$, and the surface of the shock front. We call these surfaces the tail and front of the wave, respectively. The wave element, included in the examined cylinder, in the first approximation, in entirety, is carried along the beam with a velocity q_0. By virtue of formula (1.4), the rate of propagation of its tail along the normal n_{0i} to itself is equal to zero during the entire movement. The rate of propagation of the front along that same normal will be, obviously

$$\lambda n_{0j}\partial(a_0 + v_{0i}n_{0i})/\partial x_j$$

Expressing a_0 by means of Eq. (1.4) we find that the variation of the quantity λ in geometric acoustics is determined by the relationship

$$\frac{d\lambda}{dt} = -\lambda v_{0i}n_j\frac{\partial n_i}{\partial x_j}$$

Solving this equation, we have

$$\lambda = \lambda_0 \exp\left(-\int_{t_0}^{t} v_{0i}n_j\frac{\partial n_i}{\partial x_j}dt\right) \tag{1.10}$$

Here λ_0 is the initial length of the wave element, the time t is introduced by expression (1.9), but the subscript zero of the normal component is omitted.

It follows from formula (1.10) that the wavelength varies also in the approximation of geometric acoustics, if the medium is nonuniform. However, the relationship for λ, obtained in Ref. 8, when studying nonstationary processes of shock wave propagation is different from (1.10). This is a direct result of the difference just indicated in the position of the rays relative to the characteristic surfaces in steady-state and nonsteady phenomena. In turn, as will be seen later on, it leads to the difference in the corresponding laws of extinction of the excess gas parameters along the density jumps.

2. Until now, the properties of shock waves have been examined in the framework of geometrical acoustics. Let us employ previous results to explain the properties of waves with a small amplitude in the second approximation. According to the previous results, the structure of the perturbed region in a steady-state flow will be examined from the point of view of nonsteady-state motions.

First of all, we note that formulas (1.6), which determine the relationships between the values p', ρ', a', and v', correspond to flows of the ordinary wave type.[12] Therefore, if there is a diverging shock wave with a small width of perturbed region in the nonuniform supersonic flow, each of its small elements can be examined as a plane Rieman wave that is carried laterally by a uniform flow with velocity q_0. When discontinuities are present, the Rieman solution loses significance; however, for waves of small amplitude with an accuracy up to the terms of a second order relative to p'/p_0, the wave remains ordinary.

Let us examine the movement of the wave element in more detail, whereupon, for the sake of simplicity, we shall consider that all the excess values of the perturbed region, bounded by the shock front, have a triangular profile. The velocity U of the phase translation of the profile in the Rieman solution is equal to[12]

$$U = v_0 + a_0 + m_0(p'/\rho_0 a_0) \tag{2.1}$$

where v_0 is the velocity of the particles in a perturbed flow. According to equality (2.1), the points with large excess pressures in time pass the points which have less excess pressures. The solution ceases to be single valued. Difficulty is removed by introducing a surface of discontinuity. Its location is determined by a simple geometric condition that states that the area under the curve describing the profile of the wave element remains the same as that for the multivalued curve determined by the Rieman solution.[1]

Let us derive an equation according to which the distance l' varies, which separates the points with a zero excess pressure and the points corresponding to the maximum excess pressure in a continuous three-valued solution. As in the first-order theory just examined, the rate of propagation of the tail of the wave element, where $p' = 0$, along the normal n_{0i} to itself, remains equal to zero during the entire movement. The rate of propagation of the points with a maximum excess pressure along that same normal is given by

$$l' n_{0j} \frac{\partial(a_0 + v_{0i} n_{0i})}{\partial x_j} + m_0 \frac{p'}{\rho_0 a_0}$$

Expressing a_0 with Eq. (1.4), we then have

$$\frac{dl'}{dt} = -l' v_{0i} n_j \frac{\partial n_i}{\partial x_j} + m_0 \frac{p'}{\rho_0 a_0} \tag{2.2}$$

We will write its solution in the form

$$l' = \lambda + \exp\left(-\int_{t_0}^{t} v_{0i} n_j \frac{\partial n_i}{\partial x_j}\, d\tau\right) \times \int_{t_0}^{t} \left[\frac{m_0}{\rho_0\, a_0} \exp\left(\int_{t_0}^{\tau} v_{0i}\, n_j \frac{\partial n_i}{\partial x_j}\, d\tau'\right) p'\right] d\tau \tag{2.3}$$

Here the time t is introduced by Eq. (1.9), the value λ by Eq. (1.10), and the zero subscript of the component of the normal, as before, is omitted.

Formulas (2.2) and (2.3) indicate that the propagation of the density jumps in nonuniform supersonic flows is accompanied by a peculiar superposition of the acoustical effect of the wavelength variation when passing through the layers with different unperturbed parameters and a phenomenon caused by an effect of a higher order with respect to amplitude.[9] In geometric acoustics, the values l' and λ coincide.

Using the geometric condition just stated, which determines the position of the shock front in Rieman solution, it is now easy to obtain expressions for the width of the region of perturbed flow λ_* and of excess pressure at the shock front p_*' in the higher than the geometric acoustical approximation. We have

$$\lambda_* = \lambda \left\{1 + \frac{p_0'}{\lambda_0 \sqrt{\rho_{00}\, a_{00}}} \int_{t_0}^{t} \left[\frac{m_0}{\sqrt{\rho_0 a_0}} \frac{1}{L} \times \exp\left(\int_{t_0}^{\tau} v_{0i} n_j \frac{\partial n_i}{\partial x_j}\, d\tau'\right)\right] d\tau\right\}^{1/2} \tag{2.4}$$

$$p_*' = p_0' \frac{1}{L} \sqrt{\frac{\rho_0 a_0}{\rho_{00} a_{00}}} \left\{1 + \frac{p_0'}{\lambda_0 \sqrt{\rho_{00} a_{00}}} \int_{t_0}^{t} \left[\frac{m_0}{\sqrt{\rho_0 a_0}} \frac{1}{L} \times \exp\left(\int_{t_0}^{\tau} v_{0i} n_j \frac{\partial n_i}{\partial x_j}\, d\tau'\right)\right] d\tau\right\}^{-1/2} \tag{2.5}$$

$$dt = dl/\sqrt{v_0^2 - a_0^2}$$

It should be noted that formulas (2.4) and (2.5) differ from the corresponding equalities given in Ref. 9 and those that determine the laws of variation of wavelengths and excess pressure at the shock front of nonstationary waves. The reason for this, as already stated, is explained by the fact that, in geometrical acoustics, the rays in nonstationary phenomena intersect the region of perturbed flow in physical space (x_i), whereas, in steady-state phenomena, they are entirely propagated within it, never passing beyond its boundaries.

3. Consider in particular the case of plane-parallel flows, which has great practical significance.

The equality

$$d\mathbf{n} = -\boldsymbol{\tau} d(\alpha_0 + \vartheta_0) \tag{3.1}$$

is correct for the increment of the unit vector of the normal **n** to the characteristic curve when passing between two infinitely close characteristics.

Here $\boldsymbol{\tau}$ is the unit vector of the tangent to the characteristic curve, $\alpha_0 = \text{arc}\sin(a_0/v_0)$ is the Mach angle, and ϑ_0 the angle of inclination of the vector velocity with respect to a certain fixed direction. Let this direction be taken along the x_1 axis; the direction perpendicular to it, with axis x_2. From formula (3.1), it follows that

$$v_{0i} dn_i = -v_{0i} \tau_i d(\alpha_0 + \vartheta_0)$$

Whence we obtain

$$-v_{0i} n_j \frac{\partial n_i}{\partial x_j} = v_0 \cos\alpha_0 n_j \frac{\partial(\alpha_0 + \vartheta_0)}{\partial x_j} \tag{3.2}$$

where $n_1 = -\sin(\alpha_0 + \vartheta_0)$, $n_2 = \cos(\alpha_0 + \vartheta_0)$. Employing within the integral (1.10) the variable l rather than t, we have the expression for λ in the form

$$\lambda = \lambda_0 \exp\left[\int_{l_0}^{l} n_j \frac{\partial(\alpha_0 + \vartheta_0)}{\partial x_j}\, dl\right] \tag{3.3}$$

Let us transform Eq. (1.8) giving the value of the coefficient L. The first two terms in it are calculated without difficulty using the values α_0, v_0, and ϑ_0. Actually, having designated $\theta_{01} = \cos\vartheta_0$ and $\theta_{02} = \sin\vartheta_0$, we will get

$$\frac{\partial v_{0i}}{\partial x_i} = \theta_{0i} \frac{\partial v_0}{\partial x_i} + v_0 \frac{\partial \theta_{0i}}{\partial x_i} \tag{3.4}$$

The expression for the last term can be simplified somewhat. For this reason we shall differentiate relationship (1.4):

$$n_i dv_{0i} = v_{0i} dn_i - da_0$$

Hence we will get, considering Eq. (3.1):

$$n_i dv_{0i} = v_0 \cos\alpha_0 d\vartheta_0 - \sin\alpha_0 dv_0$$

We will represent the last term in the expression for L in the form

$$n_i n_j \frac{\partial v_{0i}}{\partial x_j} = v_0 \cos\alpha_0 n_j \frac{\partial \vartheta_0}{\partial x_j} - \sin\alpha_0 n_j \frac{\partial v_0}{\partial x_j} \tag{3.5}$$

Combining formulas (3.4) and (3.5) and using variable l in the integral (1.8), we have

$$L = \exp\left\{\frac{1}{2}\int_{l_0}^{l}\left[a_0 \frac{\partial n_j}{\partial x_j} + k_0 v_0 \frac{\partial \theta_{0j}}{\partial x_j} + v_0 \cos\alpha_0 n_j \frac{\partial \vartheta_0}{\partial x_j} + (k_0\theta_{0j} - \sin\alpha_0 n_j)\frac{\partial v_0}{\partial x_j}\right]\frac{dl}{v_0 \cos\alpha_0}\right\} \tag{3.6}$$

Equations (2.4) and (2.5) take on the form

$$\lambda_* = \lambda\left\{1 + \frac{p_0'}{\lambda_0\sqrt{\rho_{00}a_{00}}}\int_{l_0}^{l}\left[\frac{m_0}{\sqrt{\rho_0 a_0}}\frac{1}{L}\times \exp\left(-\int_{l_0}^{\sigma} n_i \frac{\partial(\alpha_0+\vartheta_0)}{\partial x_i}d\sigma'\right)\right]\frac{d\sigma}{v_0\cos\alpha_0}\right\}^{-1/2} \tag{3.7}$$

$$p_*' = p_0'\frac{1}{L}\sqrt{\frac{\rho_0 a_0}{\rho_{00}a_{00}}}\times\left\{1 + \frac{p_0'}{\lambda_0\sqrt{\rho_{00}a_{00}}}\int_{l_0}^{l}\left[\frac{m_0}{\sqrt{\rho_0 a_0}}\frac{1}{L}\times \exp\left(-\int_{l_0}^{\sigma} n_i \frac{\partial(\alpha_0+\vartheta_0)}{\partial x_i}d\sigma'\right)\right]\frac{d\tau}{v_0\cos\alpha_0}\right\}^{-1/2} \tag{3.8}$$

Here the values λ and L are given by Eqs. (3.3) and (3.6). If the incident flow is homogeneous, $L = 1$. Considering that it is directed along the x_1 axis, from Eqs. (3.7) and (3.8) we will get the asymptotic laws of extinction of shock waves, which were established by Landau[1] and Whitham[10]:

$$\lambda_* = \lambda_0\left[1 + \frac{m_0 p_0'(x_2 - x_{20})}{\rho_0 b_0 v_0^2 \sin^3\alpha_0 \cos\alpha_0}\right]^{1/2}$$

$$p_*' = p_0'\left[1 + \frac{m_0 p_0'(x_2 - x_{20})}{\rho_0 b_0 v_0^2 \sin^3\alpha_0 \cos\alpha_0}\right]^{-1/2}$$

Here $b_0 = \lambda_0/\sin\alpha_0$ is the initial distance along the x_1 axis between the density jump and the characteristic which has the zero value of the excess pressure. If the shock wave is produced by a thin profile set at a small angle of attack in a supersonic flow, then the quantity b_0 is equal to a part of the chord of the profile from the tip to the point where the tangent to its generatrix is parallel to the vector velocity of the incident flow.

Let us examine the flow around a body of revolution at a zero angle of attack in a uniform supersonic flow. In this case

$$L = \exp\left[\frac{1}{2}\int_{l_0}^{l}\frac{1}{r}\frac{a_0}{v_0}\frac{dl}{\cos\alpha_0}\right]$$

Here r is the distance measured from the axis of symmetry x_1. Since $dl = \cot\alpha_0\, dr$, the expression for L takes on the form

$$L = \sqrt{r/r_0}$$

Taking the equality obtained into account, from relationships (2.4) and (2.5) we can easily introduce the asymptotic laws of extinction of shock waves in flows with axial symmetry (1.10):

$$\lambda_* = \lambda_0\left[1 + \frac{2m_0 p_0'\sqrt{r_0}(\sqrt{r} - \sqrt{r_0})}{\rho_0 b_0 v_0^2 \sin^3\alpha_0 \cos\alpha_0}\right]^{1/2}$$

$$p_*' = p_0'\sqrt{\frac{r_0}{r}}\left[1 + \frac{2m_0 p_0'\sqrt{r_0}(\sqrt{r} - \sqrt{r_0})}{\rho_0 b_0 v_0^2 \sin^3\alpha_0 \cos\alpha_0}\right]^{-1/2}$$

4. For plane-parallel flows, we shall show that the stated results, based on a conception of a small element of a perturbed region of flow having been established as a Rieman wave, coincide with the results which we can obtain by ordinary methods. Actually, in geometrical acoustics, the increment $d\lambda$ of the width of the region of perturbed movement is determined by the expression

$$d\lambda = \lambda n_i \frac{\partial(\alpha_0 + \vartheta_0)}{\partial x_i}dl \tag{4.1}$$

which leads to formula (3.3).

Furthermore, any small element of the perturbed region can be considered as a Prandtl-Meyer flow, bounded by the shock wave. But the connection between the gas parameters in Rieman and Prandtl-Meyer solutions, as indicated by A. A. Nikol'skii[13] is the same in the first approximation, if we identify the velocity of the particles in the first solution with the components of the velocity with respect to the normal to the characteristic in the second.

It is easy to show that this connection remains the same with an accuracy to the terms of the second order relative to p'/p_0. It is also easy to show that in the case when the gas parameters in a Prandtl-Meyer flow vary little, the velocity component along the characteristic in the first approximation will be the same for the entire flow. Thus, any small region of such flow may be considered as an elementary Rieman wave carried laterally by a uniform flow. It remains to be shown that the angles of inclination of the characteristics in Prandtl-Meyer flow and the location of the density jump are described correctly using the considerations introduced from the view-point of nonstationary movements.

In Prandtl-Meyer flow, the difference $(\alpha' + \vartheta')$ between the angle of inclination of the characteristics which have the pressure p', and the angle of inclination of the characteristic with $p' = 0$ is given by the expression[12]

$$\alpha' + \vartheta' = m_0 \tan\alpha_0(p'/\rho_0 a_0^2) \tag{4.2}$$

The rate of the phase propagation U in a Rieman wave running against the flow, which has a sound velocity, according to formula (2.1) is

$$U = m_0(p'/\rho_0 a_0)$$

Having divided this expression by the value of the ray velocity $q_0 = a_0/\tan\alpha_0$, which coincides with the velocity of the lateral drift of the wave, we will get, obviously, the increment of the angle of inclination of the characteristic with Prandtl-Meyer flow. It agrees with Eq. (4.2).

With similar considerations we can also introduce a formula giving the difference β' between the angle of inclination of the shock wave and the angle of inclination of the characteristic in an unperturbed flow. For steady-state gas movements, we have[12]

$$\beta' = \tfrac{1}{2}m_0 \tan\alpha_0(p'/\rho_0 a_0^2) \tag{4.3}$$

The rate of propagation N of a shock wave in the direction against a sonic flow is determined by

$$N = \tfrac{1}{2}m_0(p'/\rho_0 a_0)$$

In order to obtain the deviation of the angle of inclination of the density jump β' relative to the characteristic with a zero value of the excess pressure, it is necessary, as before, to divide the last expression by $q_0 = a_0/\tan\alpha_0$. As a result, we will arrive at Eq. (4.3), by which the proof is completed.

It is evident in the example of plane-parallel flows that both approaches to the problem of the laws of the development of density jumps in steady-state supersonic flows are the same and lead to the same results. However, study of a general case of spatial gas motions is considerably simpler if it is carried out from the point of view of nonstationary motions, since flow in a Rieman wave has only one dimension.

In conclusion, let us note that the theory that has been developed is correct so long as

$$R,H \gg \lambda_* = \lambda \left\{1 + \frac{p_0'}{\lambda_0\sqrt{\rho_{00}\,a_{00}}} \int_{t_0}^{t} \left[\frac{m_0}{\sqrt{\rho_0 a_0}}\frac{1}{L} \times \exp\left(\int_{t_0}^{\tau} v_{0i} n_j \frac{\partial n_i}{\partial x_j}\, d\tau'\right)\right] d\tau\right\}^{1/2}$$

is fulfilled, which can restrict the limits of its application when $l \to \infty$. In the case when $H \gg \lambda_*$, Eqs. (2.4) and (2.5) have an asymptotic nature.

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Reaction of Producing Hydrogen Peroxide in Liquid Ammonia

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The study of reactions of peroxide compounds in liquid ammonia is particularly interesting. Peroxides in this solvent were first obtained by the action of dry oxygen on solutions of alkaline metals[1] and alkaline earth metals.[2] Afterward these reactions were studied in detail.[3–9] The reactions of the double exchange between suspensions of hyperoxides of sodium and potassium and solutions of nitrates of various metals were investigated.[9–11] In the separation of inorganic ozonides, ammonia was used as a dissolvent.[12, 13] References to the study of reactions between hydrogen peroxide and alkaline metals or their derivatives in a solution of liquid ammonia were not found in the literature.

At temperatures below $-30°$ and concentration of more than 35% by weight, hydrogen peroxide forms sufficiently stable solutions in liquid ammonia.[14] In water solution the dissociation K of H_2O_2 after the first stage is equal to $\sim 1 \times 10^{-12}$. In liquid ammonia it should increase, since affinity for the proton of the molecule of this solvent is greater than for the molecule of water. Alkaline metals, their amides, and a number of salts, such as nitrates, nitrites, perchlorates, rhodonides, and others, form stable solutions in liquid ammonia. We undertook a thermodynamic evaluation of the possibility of reactions between hydrogen peroxide and the enumerated classes of compounds (metal, amide, salt of oxygen acid) in liquid ammonia. As an example sodium and its derivatives were chosen.

Standard thermodynamic data for the substances participating in the reaction were taken from literature,[15–19] $S°$ data for $NaNH_2$ (solid) and NH_3 (liquid) were not found. The absence of $S°$ values for any amides does not permit us to estimate this quantity for $NaNH_2$; $S°NH_3$ (liquid) was assumed equal to 28 cal/deg-mole, taking into account changes of $S°$ for H_2O and H_2O_2 as a function of $\Delta H°$ at the transition from liquid into the gaseous state. Results of the calculation are given in Table 1.

The displacement of hydrogen from H_2O_2 by metallic sodium according to reaction 1 ought to be no less successful in ammonia than in ether.[20] The interaction of amide with H_2O_2 by reaction 2, corresponding to the process of neutralization in water solution, ought to proceed in the direction of creating sodium peroxide. This is confirmed by the substantially larger negative $\Delta H°$ value of the reaction, which will hardly permit us to obtain a positive $\Delta Z°$ value after accounting for the entropic term.

Judging by the magnitude of $\Delta Z°$, reaction 3 in liquid ammonia ought to proceed from right to left, that is, an accumulation of hydrogen peroxide will take place. Calculation of the magnitude $\Delta Z°$ of the reaction 3 for the temperature range $-50°$ to $+100°$C gives for $-50°$ the value +6 kcal/mole and for 100° the value +22.4 kcal/mole. The preserva-

Table 1 Evaluation of the possibility of producing hydrogen peroxide reactions in liquid ammonia

Reaction	$\Delta H°$, kcal/mole	$S°$, kcal/deg-mole	$\Delta Z°$, kcal/mole
1) $2Na(s) + H_2O_2(l) = Na_2O_2(s) + H_2(g)$	−85.8	+7.4	−83.6
2) $2NaNH_2(s) + H_2O_2(l) = Na_2O_2(s) + 2NH_3(g)$	−53.0	...	...
3) $2NaNO_3(s) + H_2O_2(l) = Na_2O_2(s) + 2NH_4NO_3(s)$	+6.6	−39.0	+18.2

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